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Simultaneous Calculation of Fourier–Bessel Transforms up to Order N*

SÉBASTIEN M. CANDEL

Office National d'Etudes et de Recherches Aérospatiales (ONERA), 92320 Chatillon, and Ecole Centrale des Arts et Manufactures, 92290 Chatenay-Malabry, France

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In many practical applications it is necessary to calculate the first N Fourier–Bessel transforms. This paper describes procedures allowing the simultaneous evaluation of these transforms. The calculation involves a basic one-dimensional Fourier transform, repeated selection of suitable Fourier components followed by successive evaluation of Fourier series coefficients. Determination of L samples of the first N Fourier–Bessel transforms requires of the order of $2LN \log_2 N$ operations and this number is comparable to the operation count corresponding to a two-dimensional discrete Fourier transform.

1. INTRODUCTION

Fourier–Bessel transforms, also called Hankel transforms, are useful tools of mathematical physics and signal processing. They are of considerable importance in the analysis of optical systems and in laser beam propagation problems. They are extensively used in studies concerned with waves in layered media and are currently applied in image processing and seismic data analysis. In many practical situations it is desired to calculate a complete set of Fourier–Bessel transforms of a single function, and in some cases of technological importance the number N of transforms to be determined is large.

This paper describes a method allowing the simultaneous calculation of such transform sets. The method is ideally suited to cases where N is a large number and a power of 2. In such circumstances the method provides L samples of the first N Fourier–Bessel transforms while it requires of the order of $2LN \log_2 N$ operations. This number is comparable to that corresponding to a two-dimensional fast Fourier transform of a sequence of NL samples and in this sense the algorithms described in the present paper may be consider to be "fast."

At this point it is worth reviewing previous work concerned with the calculation of Fourier–Bessel transforms. An early and elegant proposal due to Siegman [1] is based on the "Gardner transform." Argument and integration variables are first

* The main part of this work was performed at the Joint Institute for Aeronautics and Acoustics (JIAA), Department of Aeronautics, Stanford University, Stanford, Calif. 94305.

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replaced by exponential variables and the Fourier–Bessel transform is thus converted into a correlation integral which may be evaluated by performing three FFT operations. Siegman's method does not provide transform samples corresponding to low values of the transform argument and a "lower end correlation" is required. Another drawback of this method is related to the exponential spacing used in sampling the function under transformation. The function must be oversampled in the range corresponding to low argument values. Despite these difficulties Siegman's method is quite valuable and it has been successfully applied in the analysis of laser systems by Sheng [2]. Extensions of Siegman's idea are also presented by Talman [3] in the calculation of several integral transforms.

Another method of interest is that developed by Cavanagh and Cook [4]. The function under transformation is expanded into Gauss–Laguerre polynomials which have known Fourier–Bessel transforms. Unfortunately practical applications require that a large number of terms be included in the transformed series for convergence.

A series expansion method is also used by Nachamkin and Maggiore [5] to calculate the magnetic field of solenoids. Oppenheim *et al.* [6] base their method on the "projection slice theorem." They show that the kth-order Fourier–Bessel integral may be deduced from the one-dimensional Fourier transform of the "projection" $p(x)$ of the function under transformation onto the real x axis. Numerical implementation is not considered in this reference and there are no test calculations. While the present paper was being prepared, a more detailed description of this technique was published by the same authors [7]. Several procedures are derived for computing the Fourier–Bessel transform which all have in common the "projection slice theorem." Only one method is explored in detail and convincing examples are given for the zeroth-order transform. The method of Oppenheim *et al.* has some features in common with that developed independently by the present author [8]. However, derivation, implementation and applications are quite distinct.

The Fourier–Bessel transform is calculated in Ref. [8] by performing a one-dimensional Fourier transformation, selecting suitable Fourier components and adding these components. This algorithm gives accurate results and is probably slightly more efficient than that of Ref. [7]. Considerable improvement of efficiency is obtained in a second paper [9], in which calculation of the Fourier–Bessel transform is based on "dual" procedures. The computation involves two complementary algorithms. The first is derived in Ref. [8] and provides the lower-order components. The second operates asymptotically and yields intermediate and higher-order components. Switching from the first to the second algorithm occurs when results of both coincide to a certain acceptable error. Such dual procedures involve a number of operations, which is of the order of that required by three to five FFT. As a consequence they are comparable in efficiency to Siegman's method. Dual procedures may be developed for computing transforms of any order but in cases where one wishes to calculate the first N transforms of a given function it is more appropriate to use the technique that we are now going to derive.

The mathematical basis of our method is formulated in Section 2. The fundamental expressions obtained are cast in discrete form in Section 3 and numerical implementation is also discussed. Section 4 contains results of test calculations and illustrates some features of the proposed algorithms.

2. Mathematical Development

The Fourier–Bessel transform of order k may be defined as:

$$F_k(r) = \int_0^\infty f(\zeta)\, \zeta J_k(\zeta r)\, d\zeta. \tag{1}$$

The kernel of this transform is the Bessel function J_k with argument ζr. The fundamental expressions used in this paper to calculate the first N Fourier–Bessel transforms may be derived from the following generating function expansion:

$$\exp[\tfrac{1}{2}z(t - 1/t)] = \sum_{k=-\infty}^{+\infty} t^k J_k(z). \tag{2}$$

This relation is proved in most special functions textbooks by expanding the left-hand side in powers of t and showing that successive coefficients of t^k are Bessel functions $J_k(z)$ (see, for example [10]). Expansion (2) yields a useful form if we substitute $t = e^{i\theta}$:

$$e^{iz\sin\theta} = \sum_{k=-\infty}^{+\infty} e^{ik\theta} J_k(z). \tag{3}$$

Now let us replace z by ζr, multiply both sides of (3) by $f(\zeta)\,\zeta$ and integrate from 0 to ∞:

$$\int_0^\infty e^{i\zeta r\sin\theta} f(\zeta)\, \zeta\, d\zeta = \int_0^\infty \sum_{k=-\infty}^{+\infty} e^{ik\theta} J_k(\zeta r) f(\zeta)\, \zeta\, d\zeta. \tag{4}$$

The Fourier–Bessel transforms of $f(\zeta)$ are made apparent by interchanging integration and summation operations:

$$\int_0^\infty e^{i\zeta r\sin\theta} f(\zeta)\, \zeta\, d\zeta = \sum_{k=-\infty}^{+\infty} e^{ik\theta} F_k(r). \tag{5}$$

$F_k(r)$ represents the kth-order Fourier–Bessel transform as defined by expression (1). Now let $\phi(\eta)$ designate the one-sided Fourier transform of $f(\zeta)\,\zeta$:

$$\phi(\eta) = \int_0^\infty e^{i\zeta\eta} f(\zeta)\, \zeta\, d\zeta; \tag{6}$$

then expression (5) becomes

$$\phi(r \sin \theta) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} F_k(r) \tag{7}$$

The function $\phi(r \sin \theta)$ is periodic with a period of 2π and its complex Fourier-series expansion forms the right-hand side of (7). Thus the Fourier-series coefficients may be obtained from

$$F_k(r) = \frac{1}{2\pi} \int_0^{2\pi} \phi(r \sin \theta)\, e^{-ik\theta}\, d\theta. \tag{8}$$

Clearly the Fourier-Bessel transforms may be calculated as the Fourier series coefficients of $\phi(r \sin \theta)$. Expressions (6) and (8) define the basic algorithm developed in the present paper.

In some circumstances it is convenient and more efficient to use a variant of expression (8). Observing that $\phi(r \sin \theta)$ is an even function with respect to $\theta = \pi/2$ it is a simple matter to show that

$$F_k(r) = G_k(r) + (-1)^k\, G_{-k}(r), \tag{9}$$

where

$$G_k(r) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \phi(r \sin \theta)\, e^{-ik\theta}\, d\theta. \tag{10}$$

Expressions (6), (9) and (10) are most useful in the determination of even or odd transforms. When used in place of (6) and (8), they slightly improve the computation efficiency.

3. Numerical Implementation

A. *Basic Algorithms*

Equations (6) and (8) or (6), (9), (10) form the basis of the class of numerical algorithms described in this section. For practical application it is essential to discretize the functions and integrals appearing in these relations.

Let $\Delta\zeta$ designate the sampling period and let $f(n)$ represent the following discrete sequence:

$$\begin{aligned} \hat{f}(n) &= f(n\,\Delta\zeta), && \text{for } n = 0,\ldots, M/2 - 1, \\ &= 0, && \text{for } n = M/2,\ldots, M-1. \end{aligned} \tag{11}$$

The discrete Fourier transform of $\hat{f}(n)\, n$

$$\hat{\phi}(m) = \sum_{n=0}^{M-1} \hat{f}(n)\, n e^{i2\pi mn/M}, \qquad \text{for} \quad m = 0,\dots, M-1, \tag{12}$$

may be evaluated by performing a one-dimensional FFT. Now let $\tilde{\phi}(m)$ represent the continuous transform (6) sampled at a constant rate $1/\Delta\eta$ and scaled by $1/(\Delta\zeta)^2$:

$$\tilde{\phi}(m) = \phi(m\Delta\eta)/(\Delta\zeta)^2. \tag{13}$$

If $\Delta\eta$ and $\Delta\zeta$ satisfy the standard compatibility rule $\Delta\eta\,\Delta\zeta = 2\pi/M$, the one-sided Fourier transform $\tilde{\phi}(m)$ may be estimated by the discrete Fourier transform (12):

$$\tilde{\phi}(m) \simeq \hat{\phi}(m). \tag{14}$$

We now calculate the Fourier Bessel transforms $F_k(r)$ by making use of expression (8). Actually we shall seek estimates of these transforms for a set of discrete values of r:

$$r_l = l\Delta r, \qquad l = 0, 1,\dots, L-1.$$

It is convenient (but not essential) to relate the sampling period Δr to $\Delta\zeta$ by $\Delta r\,\Delta\zeta = 2\pi/N$, where N is an integer smaller or equal to M. In these circumstances Δr is also related to the sampling period used in the Fourier transform (13):

$$\Delta r = (M/N)\,\Delta\eta. \tag{15}$$

It is now possible to generate a sampled and scaled Fourier–Bessel transform:

$$\tilde{F}_k(l) = F_k(l\Delta r)/(\Delta\zeta)^2, \qquad l = 0, 1,\dots, L-1, \tag{16}$$

and these sampled values may be estimated by replacing the continuous integral of expression (8) by a discrete Fourier transform:

$$\hat{F}_k(l) = \frac{1}{S} \sum_{j=0}^{S-1} \frac{\phi(l\Delta r \sin\theta_j)}{(\Delta\zeta)^2}\, e^{ikj2\pi/S}, \tag{17}$$

where $\theta_j = j2\pi/S$, $j = 0, 1,\dots, S-1$ designate S angles equally spaced on the $[0, 2\pi]$ interval. S is an integer, preferably a power of 2, its value being determined by consideration of accuracy and computation time. Typically S and M should be of the same order of magnitude.

It is now necessary to relate $\phi(l\Delta r \sin\theta_j)/(\Delta\zeta)^2$ to the available estimates of the one sided Fourier transform $\hat{\phi}(m)$. There are at least two simple ways for doing this:

(a) nearest-neighbour interpolation,

(b) linear interpolation.

B. *Determination of* $\phi(l\Delta r \sin\theta_j)/(\Delta\zeta)^2$ *by nearest-neigbour interpolation*

In the first method we seek the Fourier component $\hat{\phi}[m_c(j,l)]$ corresponding to a transform argument whose value is closest to $l\Delta r \sin\theta_j$. To perform this selection it is necessary to distinguish two cases.

1. *The Fourier argument* $l\Delta r \sin\theta_j$ *is positive or vanishes.* In this situation $0 \leqslant j \leqslant S/2$. Here we seek an integer $m_c(j,l)$ such that $m_c(j,l)\,\Delta\eta$ stands as the best approximation to $l\Delta r \sin\theta_j$.

Now let us define the following sequence:

$$d(j,l) = l(\Delta r/\Delta\eta)\sin\theta_j = l(M/N)\sin\theta_j, \qquad 0 \leqslant j \leqslant S/2. \tag{18}$$

Then it is a simple matter to show that $m_c(j,l)$ may be obtained from

$$m_c(j,l) = \mathrm{Int}[d(j,l) + \tfrac{1}{2}] \tag{19}$$

where Int(·) designates the integer part of its argument.

2. *The Fourier argument* $l\Delta r \sin\theta_j$ *is negative.* In this circumstance $S/2 + 1 \leqslant j \leqslant S - 1$ and it is worth observing that

$$\sin\theta_j = -\sin\theta_{S-j} \qquad \text{for} \quad S/2 + 1 \leqslant j \leqslant S - 1.$$

We now have to find an integer $m_c(j,l)$ such that $-[M - m_c(j,l)]\,\Delta\eta$ provides the best approximation for $l\Delta r \sin\theta_j$. It is not difficult to show that

$$\begin{aligned} m_c(j,l) &= M - \mathrm{Int}[d(S-j,l) + \tfrac{1}{2}] \\ &= M - m_c(S-j,l). \end{aligned} \tag{20}$$

Expressions (19) or (20) determine $m_c(j,l)$ and the Fourier–Bessel estimates are directly obtained from:

$$\hat{F}_k(l) = \frac{1}{S}\sum_{j=0}^{S-1} \hat{\phi}[m_c(j,l)]\, e^{-ikj2\pi/S}. \tag{21}$$

It is worth noting that the computation of $\hat{F}_k(l)$ at a fixed value of l only requires that $S/2 + 1$ values of $m_c(j,l)$ be available at a time. This set of integers is in turn obtained from $d(j,l)$, $j = 0, 1, \ldots, S/2$ and as a consequence the selection of the suitable Fourier components $\hat{\phi}[m_c(j,l)]$ only requires an additional storage of $S/2 + 1$ real numbers. Further economy of storage and computation time may be achieved by noting that

$$d(j,l) = d(S/2 - j, l) \qquad \text{for} \quad j = S/4 + 1, \ldots, S/2. \tag{22}$$

Using this feature it is only necessary to calculate and store the first $S/4 + 1$ values of $d(j,l)$ corresponding to $j = 0, 1, \ldots, S/4$. Furthermore, $d(j, l+1)$ may be obtained from $d(j, l)$ by a simple recursion relation. Indeed from the definition (19) of $d(j, l)$ it is possible to deduce

$$d(j, l+1) = d(j, l) + c(j), \tag{23}$$

where $c(j) = (M/N) \sin \theta_j$.

Clearly $d(j, l+1)$ is determined from $d(j, l)$ by performing a single addition. The set $c(j)$, $j = 0, 1,..., S/4$ is initially calculated and stored and the recursion process (23) starts from $d(j, 0) = 0$, $j = 0, 1,..., S/4$.

At this point it is appropriate to discuss the choice of M. This number represents the size of the one sided Fourier transform $\hat{\phi}(m)$. Assuming a given sampling period $\Delta\zeta$ and increasing M results in a reduction of the Fourier argument sampling period $\Delta\eta$. As a consequence more Fourier components are available corresponding to the same argument interval and the selection process described above provides improved estimates of $\phi(l\Delta r \sin \theta_j)/(\Delta\zeta)^2$.

To see this more clearly consider a particular argument $l\Delta r \sin \theta_j$ and for simplicity assume that it is positive. This argument is to be approximated by $m_c(j, l)\,\Delta\eta$. In this process $l \sin \theta_j$ is replaced by $m_c(j, l)\,\Delta\eta/\Delta r = m_c(j, l)\,N/M$. The error involved is bounded by

$$|l \sin \theta_j - m_c(j, l)\,N/M| \leqslant N/2M, \tag{24}$$

and it decreases as M is increased. Thus it appears advantageous to choose a sufficiently large value of M. In this way accuracy is enhanced while computation time is increased slightly. Further indications on the choice of M, N and S are given at the end of this section.

For easy reference we have summarized the various expressions leading to the Fourier–Bessel transform in Table I.

C. *Determination of* $\phi(l\Delta r \sin \theta_j)/(\Delta\zeta)^2$ *by Linear Interpolation*

The second method for the determination of $\phi(l\Delta r \sin \theta_j)/(\Delta\zeta)^2$ from the available Fourier estimates $\hat{\phi}(m)$ is based on linear interpolation. The reasoning closely follows that used above and will not be repeated. However, it is worth noting that it is necessary to replace the set of integers $m_c(j, l)$ by

$$m_L(j, l) = \text{Int}[d(j, l)], \qquad j = 0, 1,..., S/2. \tag{25}$$

Here a particular $m_L(j, l)$ is obtained by rounding $d(j, l)$ to the nearest integer so that

$$m_L(j,l) \leqslant d(j, l) \leqslant m_L(j, l) + 1. \tag{26}$$

With this definition it is possible to write

$$\begin{aligned} &\phi(l\Delta r \sin \theta j)/(\Delta\zeta)^2 \\ &\quad \simeq \hat{\phi}[m_L(j, l)][m_L(j, l) + 1 - d(j, l)] \\ &\qquad + \hat{\phi}[m_L(j, l) + 1][d(j, l) - m_L(j, l)] \qquad \text{for} \quad j = 0, 1,..., S/2, \end{aligned} \tag{27}$$

TABLE I

1. Let $f(\zeta)$ designate the function under transformation.
2. Choose the sampling periods $\Delta\zeta$ and Δr to be such that $\Delta\zeta\,\Delta r = 2\pi/N$.
3. Define $\hat{f}(n)$ as follows:

$$\hat{f}(n) = f(n\Delta\zeta), \qquad n = 0,\ldots, M/2 - 1,$$
$$\hat{f}(n) = 0, \qquad n = M/2,\ldots, M - 1.$$

4. Choose $M = 2^{\gamma_M}$, $S = 2^{\gamma_S}$, where γ_M and γ_S designate integers.
5. Compute the one-sided discrete Fourier transform:

$$\hat{\phi}(m) = \sum_{n=0}^{M-1} \hat{f}(n)\, n e^{imn2\pi/M}.$$

6. Define the set of angles:

$$\theta_j = j\pi/S, \qquad j = 0,\ldots, S - 1.$$

7. Calculate and store

$$c(j) = (M/N) \sin\theta_j \qquad \text{for } j = 0,\ldots, S/4.$$

8. Set $d(j, 0) = 0$ for $j = 0,\ldots, S/2$.
9. Determine the new set $d(j, l)$ from the old set $d(j, l - 1)$:

$$d(j, l) = d(j, l - 1) + c(j) \qquad \text{for} \quad j = 0,\ldots, S/4,$$
$$d(j, l) = d(S/2 - j, l) \qquad \text{for} \quad j = S/4 + 1,\ldots, S/2$$

10. Obtain

$$m_c(j, l) = \mathrm{Int}[d(j, l) + \tfrac{1}{2}] \qquad \text{for} \quad j = 0,\ldots, S/2,$$
$$m_c(j, l) = M - m_c(S - j, l) \qquad \text{for} \quad j = S/2 + 1,\ldots, S - 1.$$

11. Compute the Fourier–Bessel estimates from:

$$\hat{F}_k(l) = \frac{1}{S} \sum_{j=0}^{S-1} \hat{\phi}[m_c(j, l)]\, e^{-ikj2\pi/S}.$$

12. Increment l and repeat from step 9.

and

$$\begin{aligned}\phi(l\Delta r \sin\theta_j)/(\Delta\zeta)^2 &\simeq \hat{\phi}[m_L(j, l)][m_L(S - j, l) + 1 - d(S - j, l)] \\ &\quad + \hat{\phi}[m_L(j, l) + 1][d(S - j, l) - m_L(S - j, l)] \qquad \text{for} \quad j = S/2 + 1,\ldots, S - 1, \qquad (28)\end{aligned}$$

where

$$m_L(j,l) = M - m_L(S-j,l), \qquad S/2+1 \leqslant j \leqslant S-1. \tag{29}$$

D. *Calculation of Even and Odd Transforms*

Even-order transforms may be determined with the same level of accuracy and an economy of computation time by making use of expressions (6) and (9) and (10). Consider (10), which serves as a definition for $G_k(r)$. Discrete estimates of this function are provided by:

$$\hat{G}_k(l) = \frac{1}{2V} \sum_{j=-V/2}^{V/2-1} \frac{\phi(l\Delta r \sin\theta_j)}{(\Delta\zeta)^2} e^{-ikj\pi/V}, \tag{30}$$

where $\theta_j = j\pi/V$, $j = -V/2,..., V/2-1$.

This approximation has the same level of accuracy as expression (17) if the number of terms involved in the sum is half of that used previously, i.e., if $V = S/2$.

Now, the finite sum (30) may be split in two partial sums:

$$\hat{G}_k(l) = \frac{1}{2V} \sum_{j=-V/2}^{-1} \frac{\phi(l\Delta r \sin\theta_j)}{(\Delta\zeta)^2} e^{-ijk\pi/V} + \frac{1}{2V} \sum_{j=0}^{V/2-1} \frac{\phi(l\Delta r \sin\theta_j)}{(\Delta\zeta)^2} e^{-ijk\pi/V}.$$

Replacing the summation index of the first sum by $j' = j + V$, it is a simple matter to show that $\hat{G}_k(l)$ may be cast in the following form:

$$\hat{G}_k(l) = \frac{1}{2V} \sum_{j=0}^{V-1} \Psi(j,l)\, e^{-ijk\pi/V}, \tag{31}$$

where the function $\Psi(j,l)$ is defined by

$$\begin{aligned} \Psi(j,l) &= \phi(l\Delta r \sin\theta_j)/(\Delta\zeta)^2 && \text{for} \quad j = 0, 1,..., V/2-1, \\ &= (-1)^k \phi(-l\Delta r \sin\theta_j)/(\Delta\zeta)^2 && \text{for} \quad j = V/2,..., V-1, \end{aligned}$$

with $\theta_j = j\pi/V$ for $j = 0, 1,..., V-1$.

Now the finite sum (31) takes the form of a discrete Fourier transform when k is an even number. Indeed let $k = 2p$ then

$$\hat{H}_p(l) = \hat{G}_{2p}(l) = \frac{1}{2V} \sum_{j=0}^{V-1} \Psi(j,l)\, e^{-ipj2\pi/V}, \tag{32}$$

where

$$\begin{aligned} \Psi(j,l) &= \phi(l\Delta r \sin\theta_j)/(\Delta\zeta)^2 && \text{for} \quad j = 0, 1,..., V/2-1, \\ &= \phi(-l\Delta r \sin\theta_j)/(\Delta\zeta)^2 && \text{for} \quad j = V/2,..., V-1. \end{aligned}$$

These estimates may be obtained by performing a fast Fourier transform of size $V = S/2$ and the Fourier–Bessel transforms of even order are then computed by specializing (9) to $k = 2p$:

$$\hat{F}_{2p}(l) = \hat{G}_{2p}(l) + \hat{G}_{-2p}(l) = \hat{H}_p(l) + \hat{H}_{V-p}(l).$$

Odd-order transforms may be obtained in a similar way but some additional calculations are involved. To see this let $k = 2p + 1$ then expression (31) becomes

$$\hat{H}_p(l) = \hat{G}_{2p+1}(l) = \frac{1}{2V} \sum_{j=0}^{V-1} \Psi_1(j, l)\, e^{-ipj2\pi/V}, \tag{33}$$

where the function $\Psi_1(j, l)$ is now defined as

$$\begin{aligned} \Psi_1(j, l) &= e^{-ij\pi/V} \phi(l\Delta r \sin \theta_j)/(\Delta\zeta)^2 && \text{for } j = 0, \ldots, V/2 - 1, \\ &= -e^{-ij\pi/V} \phi(-l\Delta r \sin \theta_j)/(\Delta\zeta)^2 && \text{for } j = V/2, \ldots, V - 1. \end{aligned}$$

Clearly $\hat{H}_p(l)$ is a discrete Fourier transform but the function under transformation has been linearly phase-shifted. Odd-order Fourier–Bessel transforms are then deduced from

$$\begin{aligned} \hat{F}_{2p+1}(l) &= \hat{G}_{2p+1}(l) - \hat{G}_{-(2p+1)}(l) \\ &= \hat{H}_p(l) - \hat{H}_{V-p-1}(l). \end{aligned} \tag{34}$$

E. *A variant of the Basic Algorithm*

Consider once more expression (30) and let $V = S/2$:

$$\hat{G}_k(l) = \frac{1}{S} \sum_{j=-S/4}^{S/4-1} \frac{\phi(l\Delta r \sin \theta_j)}{(\Delta\zeta)^2} e^{-ikj2\pi/S}. \tag{35}$$

This expression may be split in two partial sums.

$$\begin{aligned} \hat{G}_k(l) = {} & \frac{1}{S} \sum_{j=-S/4}^{-1} \frac{\phi(l\Delta r \sin \theta_j)}{(\Delta\zeta)^2} e^{-ijk2\pi/S} \\ & + \frac{1}{S} \sum_{j=0}^{S/4-1} \frac{\phi(l\Delta r \sin \theta_j)}{(\Delta\zeta)^2} e^{-ikj2\pi/S}. \end{aligned} \tag{36}$$

Replacing the summation index of the first sum by $j' = j + S$, it is possible to write

$$\hat{G}_k(l) = \frac{1}{S} \sum_{j=0}^{S-1} \Psi(j, l)\, e^{-ikj2\pi/S}, \tag{37}$$

where the function $\Psi(j, l)$ is now defined by

$$\begin{aligned} \Psi(j, l) &= \phi(l\Delta r \sin \theta_j)/(\Delta\zeta)^2 && \text{for } j = 0, \dots, S/4 - 1, \\ &= 0 && \text{for } j = S/4, \dots, 3S/4 - 1, \\ &= \phi(l\Delta r \sin \theta_j)/(\Delta\zeta)^2 && \text{for } j = 3S/4, \dots, S - 1, \end{aligned}$$

where $\theta_j = j2\pi/S$ for $j = 0, \dots, S - 1$.

This function contains $S/2$ zeroes and $S/2$ samples of $\phi/(\Delta\zeta)^2$.

Now $\hat{G}_k(l)$ as given by expression (36) is a discrete Fourier transform of size S. Even and odd Fourier–Bessel transforms are then directly determined from:

$$\hat{F}_k(l) = \hat{G}_k(l) + (-1)^k \hat{G}_{S-k}(l) \tag{38}$$

The computation time required by this scheme is slightly less than that used in he direct calculation of $\hat{F}_k(l)$. This is so because the sequence $\Psi(j, l)$ contains $S/2$ samples set equal to zero during the whole calculation.

F. *Guidelines for the Choice of M, N, S and L*

Certain guidelines may be given for the choice of the Fourier transform sizes M and S and of the numbers N and L on the basis of error analysis contained in Refs. [8] and [11] and numerical tests performed by us and reported in part in Ref. [11].

The size M of the intitial Fourier transform and the sampling period $\Delta\zeta$ must be chosen to obtain accurate estimates of $\hat{\phi}(m)$. This is a well-known problem and it is discussed in great detail by Brigham [12]. Note that accuracy increases as M is augmented but computation time only increases like $M \log_2 M$. Thus it is advantageous to use fairly large values of M, typically of the order of 1024.

The size S is fixed by similar considerations. The precision of the Fourier series estimates $\hat{F}_k(l)$ is determined by the choice of S. However, computation time now increases approximately like $SL \log_2 S$.

The number of samples to be determined appears in this product so that S should not be too large. Typically S may be of the order of 512.

There is a degree of freedom in the choice of N. To be consistent, N should be selected smaller or equal to the size of the initial Fourier transform: $N \leqslant M$. The other restriction on N is $N \geqslant 2L$. If this condition is not satisfied the determination of the Fourier–Bessel estimates corresponding to arguments r_l whose order is larger than $N/2$ cannot be carried through.

Further details and a discussion of accuracy may be found in Ref. [11].

4. Examples of Calculations

A small number of functions have known analytic Fourier–Bessel transforms of any order. Two such functions may be found in the tables assembled by Abramovitz and Stegun [13] and are used here to test the numerical algorithms developed in the previous section.

In the first test case the computations carried along the lines of Section 3D for $M = N = V = 512$ provided *even order transforms*. In the second test we used the method of Section 3E with $M = N = 512$ and $S = 1024$ and even and odd transforms were obtained simultaneously. The linear interpolation method of Section 3C was adopted in these calculations because it is more accurate and only slightly increases computation time.

As a first example consider the standard diffraction function:

$$f(\zeta) = \text{sinc}(b\zeta) = \sin(b\zeta)/b\zeta \tag{39}$$

Fourier–Bessel transforms of $f(\zeta)$ may be derived from expression 11.4.38 of Ref. [13]:

$$\begin{aligned} F_k(r) &= \frac{\cos \pi k/2}{b^2(1 - r^2/b^2)^{1/2}} \frac{(r/b)^k}{[1 + (1 - r^2/b^2)^{1/2}]^k} \qquad \text{for} \quad 0 \leqslant r \leqslant b, \\ &= \frac{1}{b^2(r^2/b^2 - 1)^{1/2}} \sin\,[k \arcsin\,(b/r)] \qquad \text{for} \quad b \leqslant r. \end{aligned} \tag{40}$$

These expressions are special cases of the discontinuous Weber–Schafheitlin integral.

To obtain discrete versions of (39) and (40) it is convenient to express parameter b in terms of the sampling period $\Delta\zeta$. Let $b = 1/Q\Delta\zeta$ then

$$\hat{f}(n) = \sin(n/Q)/(n/Q). \tag{41}$$

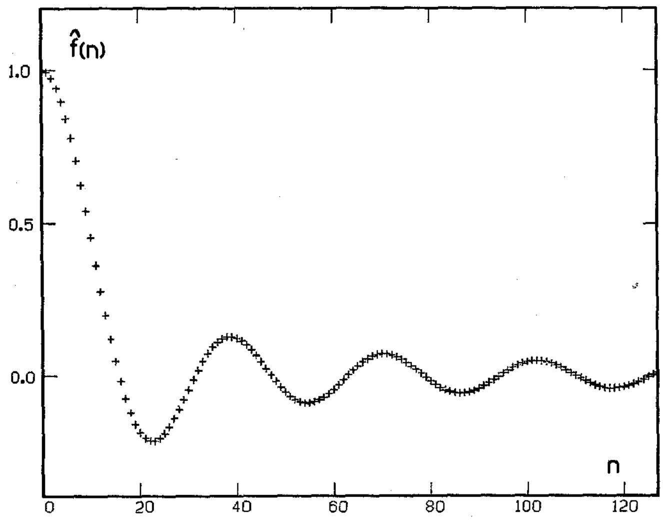
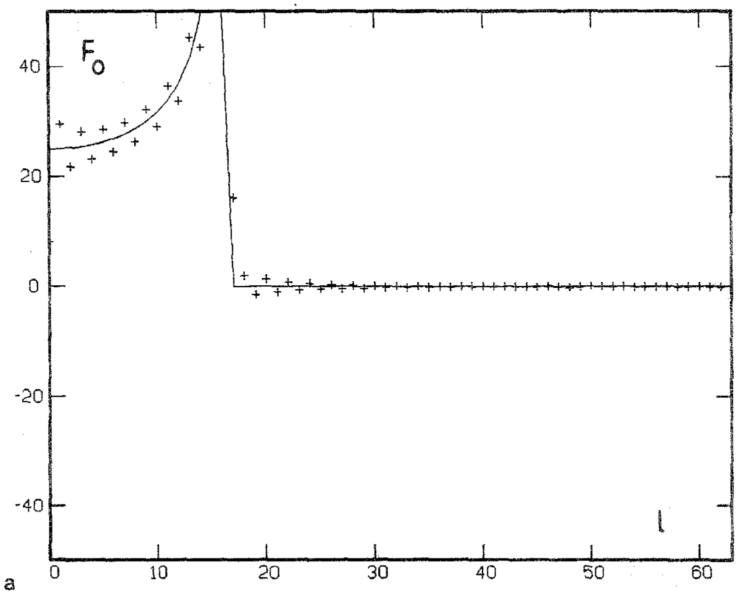


FIG. 1. Discrete samples of the sinc function $\hat{f}(n) = \sin(n/Q)/(n/Q)$, $Q = 5$.



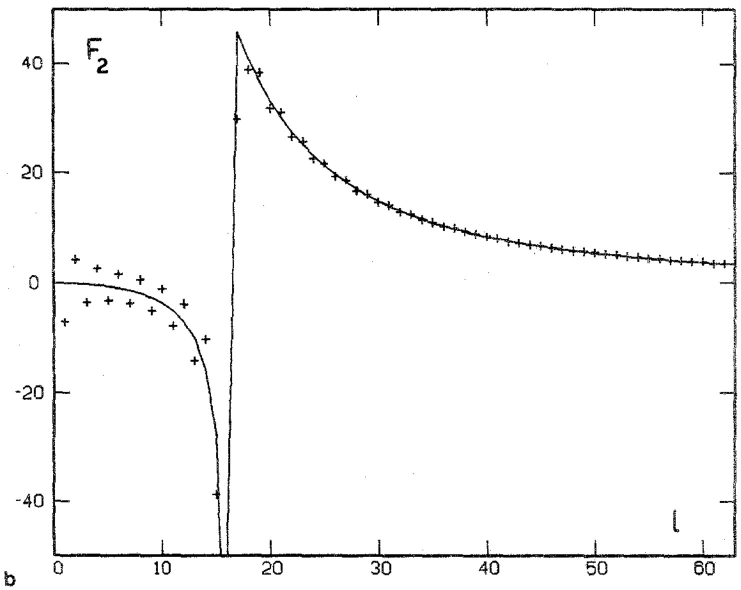
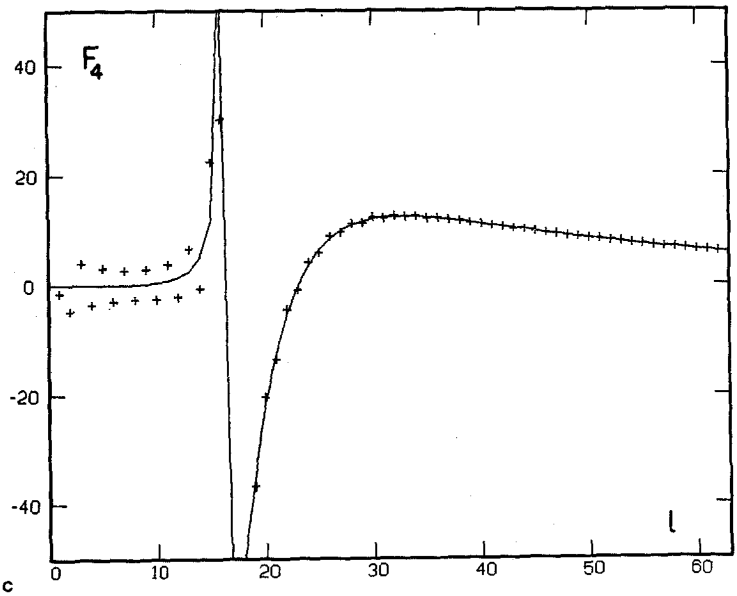
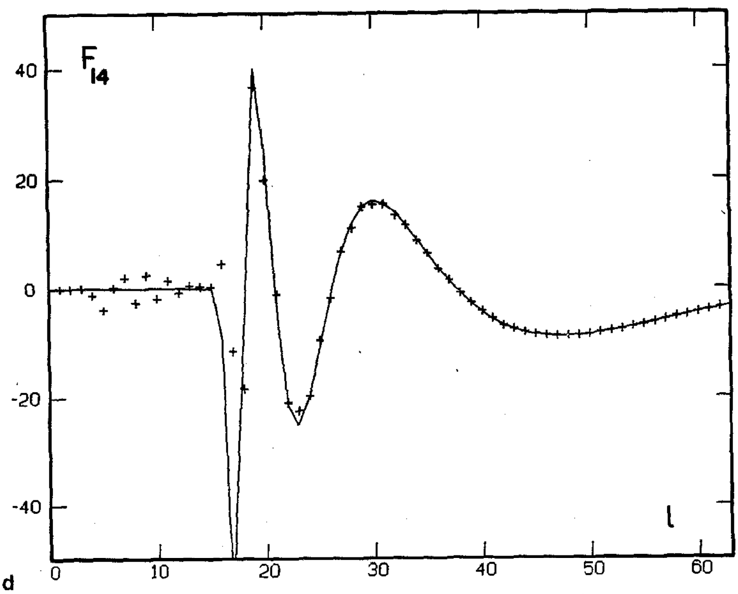


FIG. 2. Even-order Fourier–Bessel transforms of the function plotted on Fig. 1. Exact transforms are represented as solid lines. Estimates calculated by the modified algorithm of Section 3D, $M = N = V = 512$, $L = 64$. (a) $k = 0$, (b) $k = 2$, (c) $k = 4$, (d) $k = 14$.



c



d

FIG. 2–*Continued.*

The corresponding sampled and scaled Fourier–Bessel transforms are obtained from:

$$\tilde{F}_k(l) = \frac{Q^2 \cos \pi k/2}{(1-W_l^2)^{1/2}} \frac{W_l^k}{[1+(1-W_l^2)^{1/2}]^k} \qquad \text{for} \quad 0 \leqslant W_l < 1$$
$$= \frac{Q^2}{(W_l^2-1)^{1/2}} \sin[k \text{ arc} \sin(1/W_l)] \qquad \text{for} \quad 1 \leqslant W_l, \tag{42}$$

where $W_l = 2\pi Ql/N$.

The sampled function (42) is displayed in Fig. 1 and corresponding even transforms up to order 14 appear in Fig. 2 as solid lines. Numerical estimates $\hat{F}_k(l)$ are plotted on the same figure as discrete symbols.

Fourier–Bessel transforms are clearly singular. They become infinite when W_l reaches one and their jump across this point is also infinite. In such circumstances one cannot expect to have perfect agreement between exact and numerical transforms. The numerical estimates initially oscillate around the exact transform. This behavior is observed for $W_l \leqslant 1$ and it may be related to Gibb's phenomenon. As l increases and W_l becomes greater than one the error diminishes strongly and the numerical estimates nearly coincide with the exact transform.

The second test function also yields a Weber–Schafheitlin integral. Consider

$$f(\zeta) = \text{sinc}(b\zeta)/b\zeta = \sin(b\zeta)/(b\zeta)^2. \tag{43}$$

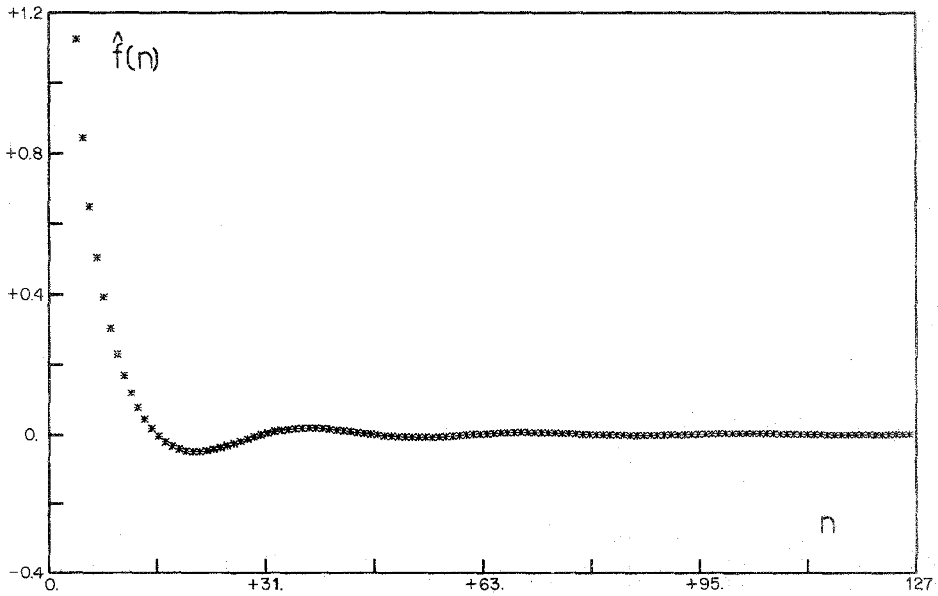


FIG. 3. Discrete samples of the function $\hat{f}(n) = \sin(n/Q)/(n/Q)^2$, $Q = 5$.

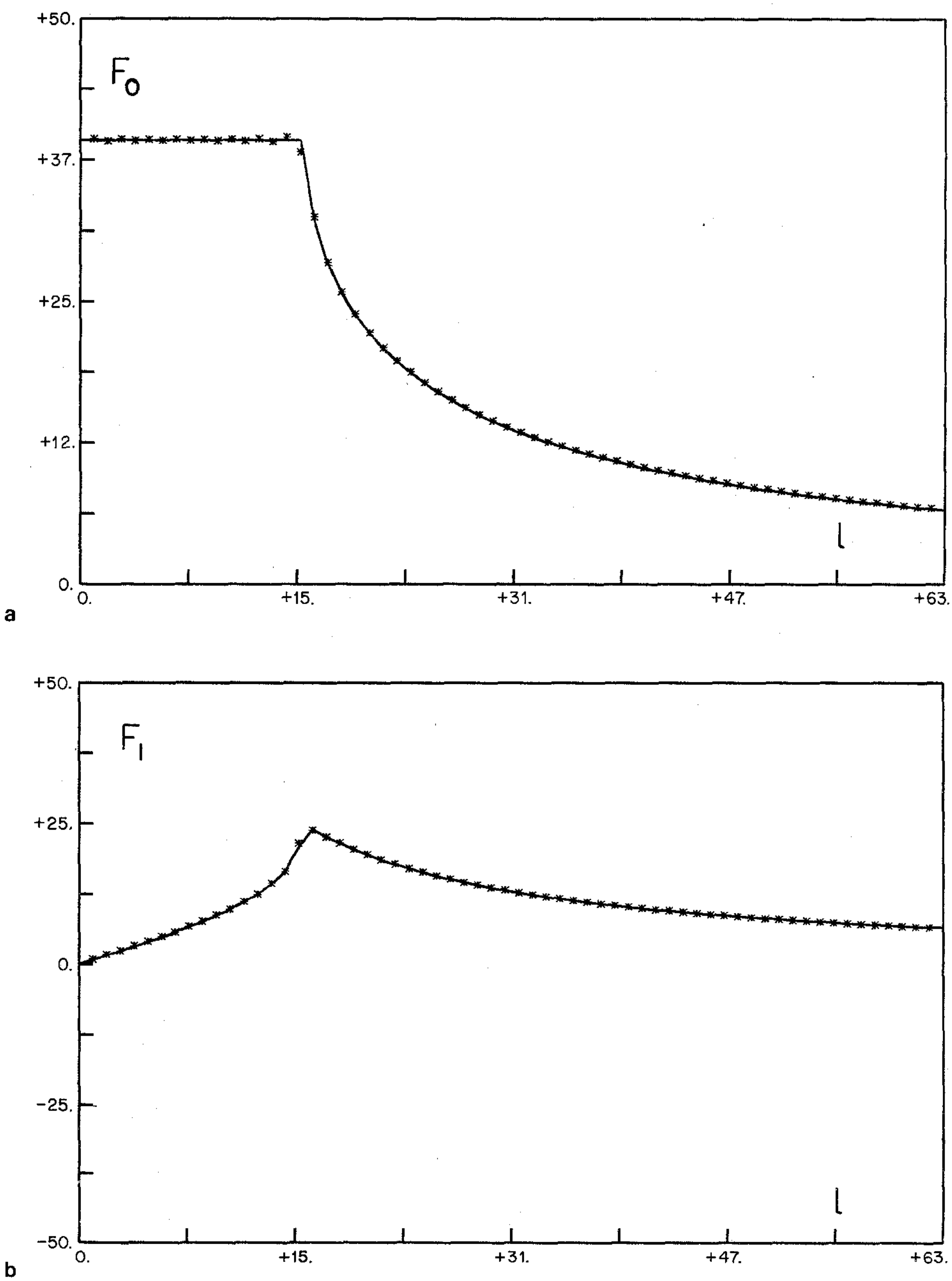
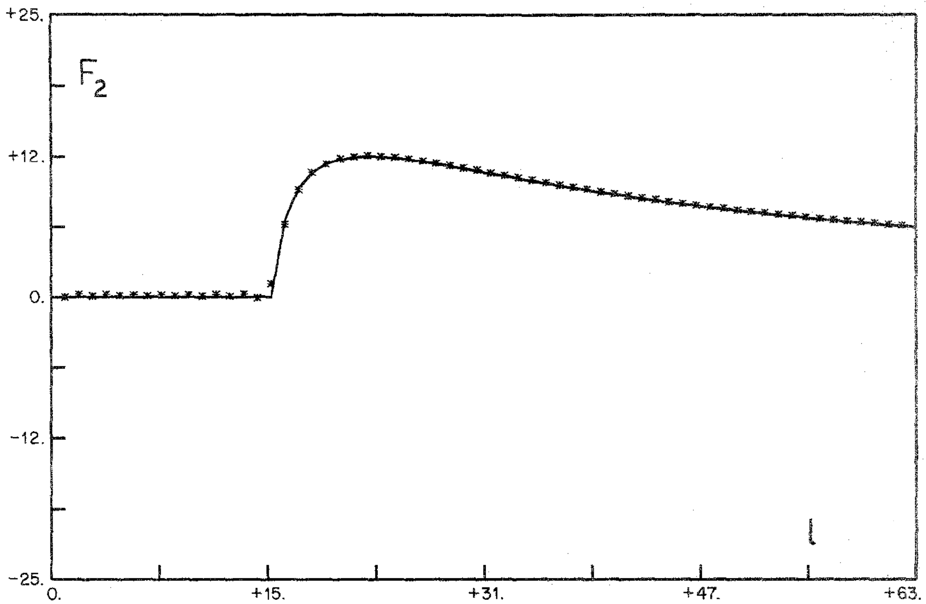
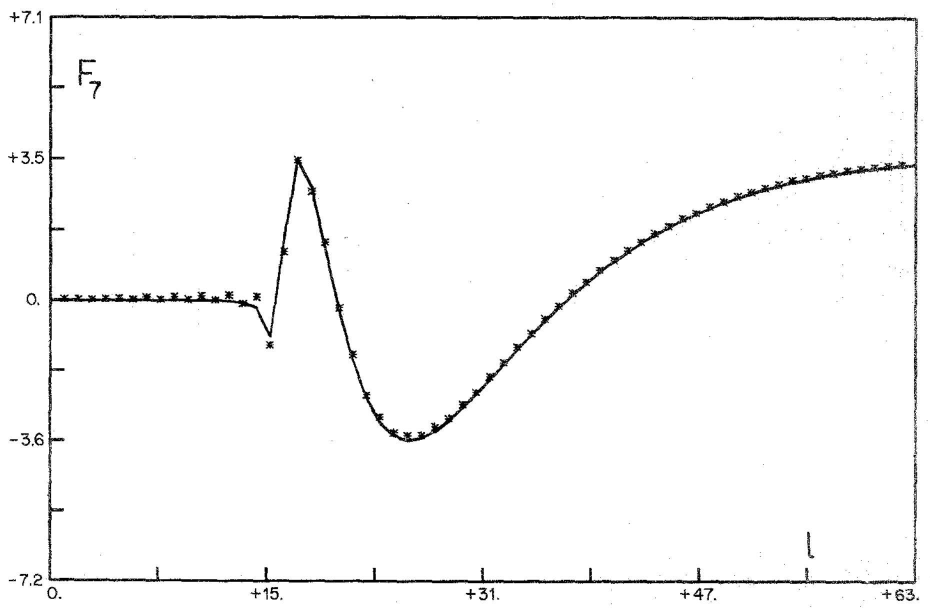


FIG. 4. Even and odd Fourier–Bessel transforms of the function plotted on Fig. 3. Exact transforms are represented as solid lines. Estimates calculated by the modified algorithm of Section 3E, $M = N = 512$, $S = 1024$, $L = 64$ (a) $k = 0$, (b) $k = 1$, (c) $k = 2$, (d) $k = 7$.



c



d

FIG. 4—*Continued.*

This function is infinite at $\zeta = 0$ (Fig. 3) but the singularity does not appear in the calculation of the Fourier–Bessel transform because $f(\zeta)$ is premultiplied by ζ.

Analytical expressions of the transforms may be deduced from expression 11.4.35 of Ref. [13]:

$$F_k(r) = \frac{1}{b^2 k} \frac{(r/b)^k}{[1 + (1 - r^2/b^2)^{1/2}]^k} \sin(\pi k/2) \qquad \text{for} \quad 0 \leqslant r \leqslant b,$$
$$= \frac{1}{b^2 k} \sin[k \arcsin(b/r)] \qquad \text{for} \quad b \leqslant r. \tag{44}$$

Sampled and scaled versions of (43) and (44) are, respectively,

$$\hat{f}(n) = \sin(n/Q)/(n/Q)^2$$

and

$$\tilde{F}_k(l) = \frac{Q^2}{k} \left[\frac{W_l}{1 + (1 - W_l)^{1/2}} \right]^k \sin \pi k/2 \qquad \text{for} \quad 0 \leqslant W_l \leqslant 1, \tag{45}$$

$$= \frac{Q^2}{k} \sin[k \arcsin(1/W_l)] \qquad \text{for} \quad 1 \leqslant W_l, \tag{46}$$

where $W_l = 2\pi Q l/N$.

The transforms are now continuous but their derivatives are not (Fig. 4). Nevertheless, the numerical estimates nearly coincide with the exact transforms and Gibb's phenomenon is in this case surprisingly weak.

In conclusion we note that in two difficult cases corresponding to functions characterized by "peculiar" transforms, the numerical algorithms described in this paper yield acceptable transform estimates at a small expense of computation time. An important advantage of the present algorithms is that they resemble the fast Fourier transform. As a consequence they may be used with equal ease and require similar precautions in sampling and windowing the function under transformation.

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